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The asymptotic behavior of solutions for a class of doubly degenerate nonlinear parabolic equations ☆

Huashui Zhan

School of Sciences, Jimei University, Xiamen 361021, PR China

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ABSTRACT

By using the Morse interaction technique, supposing that the uniqueness of the Barenblatt-type solution is true, the paper studies the large time asymptotic behavior of solutions for the doubly degenerate parabolic equation

$$u_t = \operatorname{div}\left(\left|Du^m\right|^{p-2} Du^m\right) - \left|Du^m\right|^{p_1} - u^q$$

with initial condition $u(x,0) = u_0(x)$. Here the exponents m, p, p_1, q satisfy $p > p_1 > p-1$, $q > m(p-1) > 1$, $p > 1$, $m > 1$.

1. Introduction

The objective of this paper is to study the large time asymptotic behavior of weak solutions of nonlinear parabolic equations with the following type

$$u_t = \operatorname{div}\left(\left|Du^m\right|^{p-2} Du^m\right) - \left|Du^m\right|^{p_1} - u^q \quad \text{in } S = R^N \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \quad \text{on } R^N, \tag{1.2}$$

where $p > 1$, $m > 1$, $N \geqslant 1$, $u_0(x) \in L^1(R^N)$ and D is the spatial gradient operator. The equations in the form of (1.1) have been suggested as a mathematical model for a variety of problems in mechanics, physics and biology, which can be seen in [1–3], etc. They had been widely researched, whether it is linear or nonlinear, uniformly parabolic or degenerate parabolic.

A classical example of (1.1) is the heat equation,

$$u_t = \Delta u, \tag*{$(1.1)_1$}$$

its theory is well known, among its features we find C^∞ smoothness of solutions, infinite speed of propagation of disturbances and the strong maximum principle. These properties are able to be generalized to a number of related evolution equations, notably those which are linear and uniformly parabolic. Other well-known examples of (1.1) include the porous media equation

$$u_t = \Delta u^m, \quad m > 1, \tag*{$(1.1)_2$}$$

and the evolutionary p-Laplacian equation

$$u_t = \operatorname{div}\left(|Du|^{p-2} Du\right), \quad p > 2. \tag*{$(1.1)_3$}$$

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E-mail address: hszhan@jmu.edu.cn.

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Clearly, compared with the heat equation, a marked departure occurs. These equations are degenerate parabolic and there are generally no classical solutions. Moreover, instead of the infinite speed of propagation of disturbances, the weak solutions of the Cauchy problem to $(1.1)_2$ or $(1.1)_3$ have the property of finite propagation. One can see [3,10], etc.

The existence of nonnegative solution of (1.1)–(1.2) without the absorption term $-|Du^m|^{p_1}$, defined in some weak sense, is well established (see [4] and [5]). Here we quote the following definition.

Definition 1.1. A nonnegative function $u(x,t)$ is called a weak solution of (1.1)–(1.2) if u satisfies

$$\text{(i)} \qquad u \in C\big(0,T;L^1\big(R^N\big)\big) \cap L^\infty\big(R^N \times (\tau,T)\big), \tag{1.3_1}$$

$$u^m \in L^p_{\mathrm{loc}}\big(0,T;W^{1.p}\big(R^n\big)\big), \qquad u_t \in L^1\big(R^N \times (\tau,T)\big), \quad \forall \tau > 0; \tag{1.3_2}$$

$$\text{(ii)} \qquad \iint_S \big[u\varphi_t - \big|Du^m\big|^{p-2}Du^m \cdot D\varphi - \big|Du^m\big|^{p_1}\varphi - u^q\varphi\big]\,dx\,dt = 0, \quad \forall \varphi \in C_0^1(S); \tag{1.4}$$

$$\text{(iii)} \qquad \lim_{t\to 0}\big|u(x,t) - u_0(x)\big|\,dx = 0. \tag{1.5}$$

Similarly as in [4], one is able to get the existence of the weak solution in the sense of Definition 1.1.

In this paper, we are mainly interested in the behavior of solutions as $t \to \infty$. According to the different properties of the initial function $u_0(x)$, the corresponding nonnegative solutions may have different large time asymptotic behaviors, one can refer to Refs. [6–8,12,14,15,17], etc.

It is not difficult to verify that

$$E_c = t^{\frac{-1}{\mu}}\left\{\left[b - \frac{m(p-1)-1}{mp}(N\mu)^{\frac{-1}{p-1}}\big(|x|t^{\frac{-1}{N\mu}}\big)^{\frac{p}{p-1}}\right]_+\right\}^{\frac{p-1}{m(p-1)-1}}$$

is the Barenblatt-type solution of the Cauchy problem

$$u_t = \mathrm{div}\big(\big|Du^m\big|^{p-2}Du^m\big) \quad \text{in } S = R^N \times (0,\infty), \tag{1.6}$$

$$u(x,0) = c\delta(x) \quad \text{on } R^N, \tag{1.7}$$

where $\mu = m(p-1) - 1 + \frac{p}{N}$, $c = \int_{R^N} u_0(x)\,dx$, b is a constant such that $c = \int_{R^N} E_c(x,t)\,dx$, and δ denotes the Dirac mass centered at the origin.

By assuming that the uniqueness of the Barenblatt-type solution of (1.6) is true, [12] had established the large time behavior of solutions of (1.1) with the absorption term u^q but without the absorption term $|Du^m|^{p_1}$.

If a nontrivial nonnegative function $U \in C(\overline{S} \setminus (0))$ satisfies (1.1) in the sense of distribution in S and

$$\lim_{t\to 0}\sup_{|x|>\varepsilon} U(x,t) = 0, \quad \forall \varepsilon > 0, \tag{1.8}$$

then it is called a singular solution of (1.1).

Further, if the singular solution U satisfies the following formula

$$\lim_{t\to 0}\int_{|x|\leqslant\varepsilon} U(x,t)\,dx = \infty, \quad \forall \varepsilon > 0, \tag{1.9}$$

then U is called a very singular solution of (1.1).

Let

$$u(x,t) = t^{-\alpha} f\big(|x|t^{-\beta}\big), \tag{1.10}$$

where

$$\alpha = \frac{p-p_1}{(1+m)p_1 - p}, \qquad \beta = \frac{(p_1 - p + 1)m}{(1+m)p_1 - p}.$$

Clearly, $\alpha > 0$, $\beta > 0$ if $p > p_1$, $q_1 + p_1 m > m(p-1) > 1$. If $q = \frac{mp_1}{p-p_1}$, Eq. (1.1) is equivalent to the following equation

$$\big[\big|\big(f^m\big)'\big|^{p-2}\big(f^m\big)'\big]' + \frac{n-1}{r}\big|\big(f^m\big)'\big|^{p-2}\big(f^m\big)' + \beta r f' + \alpha f - \big|\big(f^m\big)'\big|^{p_1} - f^q = 0, \tag{1.11}$$

with the initial conditions

$$f(0) = a > 0, \qquad f'(0) = 0, \tag{1.12}$$

where $r = |x|t^{-\beta}$. If a weak solution of (1.1) has the form of (1.10), as usual, it is called a self-similar solution. Recently, the author proved the existence of the similar solution of (1.1) in [17], and proved that if $N\beta < \alpha$, then $u(x,t)$ is a very singular solution of Eq. (1.1).

Now, in what follows, we assume that

$$p > p_1 > p-1, \qquad q > m(p-1) > 1. \tag{1.13}$$

By using some ideas of [4] and [12], we have the following

Theorem 1.2. *Suppose* (1.13) *is true. If E_c is a unique solution of* (1.6)–(1.7)*, then the solution u of* (1.1)–(1.2) *satisfies*

$$t^{\frac{1}{\mu}}\left|u(x,t) - E_c(x,t)\right| \to 0 \quad \text{as } t \to \infty,$$

uniformly on the sets $\{x \in R^N\colon |x| < at^{\frac{1}{\mu N}},\ a > 0\}$, where

$$c = \int_{R^N} u_0\,dx - \int_0^\infty \int_{R^N} \left|Du^m\right|^{p_1} dx\,dt - \int_0^\infty \int_{R^N} u^q\,dx\,dt.$$

Theorem 1.3. *Suppose that the solution of* (1.1)–(1.2) *is unique,* (1.13) *is true and*

$$|x|^\alpha u_0(x) \leqslant B, \qquad \lim_{|x|\to\infty} |x|^\alpha u_0(x) = C,$$

where α, B, and C are constants with $\alpha \in (0, \frac{p}{q-m(p-1)})$. Then the solution of (1.1)–(1.2) *satisfies*

$$t^{\frac{1}{q-1}} u(x,t) \to C^* \quad \text{as } t \to \infty,$$

uniformly on the sets

$$\bigl\{x \in R^N\colon |x| \leqslant at^{\frac{1}{\beta}}\bigr\}, \quad a > 0,$$

where $C^* = (\frac{1}{q-1})^{\frac{1}{q-1}}$.

Theorem 1.4. *Suppose* (1.13) *is true and*

$$|x|^\alpha u_0 \leqslant B, \qquad \alpha > \frac{p}{q-m(p-1)}, \qquad \int_{R^N} u_0(x)\,dx > 0,$$

suppose (1.1) *has a unique very singular solution U. Then the solution of* (1.1)–(1.2) *satisfies*

$$t^{\frac{1}{q-1}}\left|u(x,t) - U(x,t)\right| \to 0 \quad \text{as } t \to \infty,$$

uniformly on the sets

$$\bigl\{x \in R^N\colon |x| \leqslant at^{\frac{1}{\beta}}\bigr\}.$$

Remark 1.5. For $m = 1$, the uniqueness of solutions of (1.6)–(1.7) is known (see [16]). For $m = 1$, $p = 2$, the uniqueness of the very singular solution of (1.1) is known too (see [13]). For $p = 2$, $p_1 = 0$, if the initial value u_0 is suitably smooth, the uniqueness of the solution of (1.1) is a direct corollary of the corresponding uniqueness theorems obtained in [18–21], etc.

2. Proof of Theorem 1.2

Let u be a solution of (1.1). We define the family of functions

$$u_k = k^N u\bigl(kx, k^{N\mu} t\bigr), \quad k > 0.$$

It is easy to see that they are the solutions of the problems

$$u_t = \operatorname{div}\bigl(\left|Du^m\right|^{p-2} Du^m\bigr) - k^{\upsilon_1}\left|Du^m\right|^{p_1} - k^{\upsilon} u^q \quad \text{in } S = R^N \times (0,\infty), \tag{2.1}$$

$$u(x,0) = u_{0k}(x) \quad \text{on } R^N, \tag{2.2}$$

where $\mu = m(p-1) + \frac{p}{N} - 1$ as before and $\upsilon_1 = (Nm+1)(p-p_1) - Nm$, $\upsilon = N(q - m(p-1) - \frac{p}{N})$, $u_{0k}(x) = k^N u_0(x)$.

Lemma 2.1. *For any $s \in (0, m(p-1))$, u_k satisfies*

$$\int_0^T \int_{B_R} \frac{u_k^{s-m}}{(1+u_k^s)^2} |Du_k|^2 \, dx\, dt \leqslant c\big(s, R, |u_0|_{L^1}\big), \tag{2.3}$$

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{p}{N}-s} \, dx\, dt \leqslant c\big(s, R, |u_0|_{L^1}\big). \tag{2.4}$$

Proof. From Definition 1.1, we are able to deduce that (see [10]): for $\forall \varphi \in C^1(\bar{S})$, $\varphi = 0$ when $|x|$ is large enough, for any $t \in [0, T]$,

$$\int_{R^N} u_k \varphi(x,t)\, dx - \int_0^t \int_{R^N} \big(u_k \varphi_t - \big|Du_k^m\big|^{p-2} Du_k^m \cdot D\varphi\big)\, dx\, dt \leqslant \int_{R^N} u_{0k} \varphi(x, 0)\, dx. \tag{2.5}$$

Let

$$\psi_R \in C_0^\infty(B_{2R}), \quad 0 \leqslant \psi_R \leqslant 1, \qquad \psi_R = 1 \quad \text{on } B_R, \qquad |D\psi_R| \leqslant cR^{-1}. \tag{2.6}$$

By an approximate procedure, we can choose $\varphi = \frac{u_k^s}{1+u_k^s} \psi_R^p$ in (2.5), then

$$\begin{aligned} &\int_{R^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\, dz\, \psi_R^p\, dx + s \int_h^t \int_{R^N} \frac{u_k^{s-m}}{(1+u_k^s)^2} \big|Du_k^m\big|^p \psi_R^p\, dx\, d\tau \\ &\quad \leqslant -p \int_h^t \int_{R^N} \frac{u_k^s}{1+u_k^s} \big|Du_k^m\big|^{p-2} \psi_R^{p-1} Du_k \cdot D\psi_R\, dx\, d\tau + \int_{R^N} \int_0^{u_k(x,h)} \frac{z^s}{1+z^s}\, dz\, \psi_R^p\, dx, \end{aligned} \tag{2.7}$$

where $0 < h < t$. Noticing

$$\begin{aligned} &\left| \int_h^t \int_{R^N} \frac{u_k^s}{1+u_k^s} \big|Du_k^m\big|^{p-2} \psi_R^{p-1}(x) Du_k^m \cdot D\psi_R\, dx\, d\tau \right| \\ &\quad \leqslant \int_h^t \int_{R^N} \left\{ \varepsilon \left[\frac{u_k^{(s-m)\frac{p-1}{p}}}{(1+u_k^s)^{2\frac{p-1}{p}}} \big|Du_k^m\big|^{p-1} \psi_R^{p-1} \right]^{\frac{p}{p-1}} + c(\varepsilon) \left[\frac{u_k^{(1-s+m)\frac{p-1}{p}}}{(1+u_k^s)^{1-2\frac{p-1}{p}}} |D\psi_R| \right]^p \right\} dx\, d\tau \\ &\quad = \varepsilon \int_h^t \int_{R^N} \frac{u_k^{s-m}}{(1+u_k^s)^2} \big|Du_k^m\big|^p \psi_R^p\, dx\, d\tau + c(\varepsilon) \int_h^t \int_{R^N} u_k^{m(p-1)+s} |D\psi_R|^p\, dx\, d\tau \\ &\qquad + c(\varepsilon) \int_h^t \int_{R^N} u_k^{(m+s)(p-1)} |D\psi_R|^p\, dx\, d\tau, \end{aligned} \tag{2.8}$$

and

$$\int_{R^N} \int_0^{u_k(x,h)} \frac{z^s}{1+z^s}\, dz\, \psi_R^p\, dx \leqslant \int_{R^N} u\big(x, k^{N\mu} h\big)\, dx, \tag{2.9}$$

then by (2.7)–(2.9), we obtain

$$\begin{aligned} &\sup_{0<t<T} \int_{R^N} \int_0^{u_k(x,t)} \frac{z^s}{1+z^s}\, dz\, dx + \int_h^T \int_{R^N} \frac{u_k^{s-m}}{(1+u_k^s)^2} \big|Du_k^m\big|^p \psi_R^p\, dx\, d\tau \\ &\quad \leqslant c \int_{R^N} u\big(x, k^{N\mu} h\big)\, dx + c \int_h^T \int_{R^N} \big(u_k^{p+s+m-1} + u_k^{m(p-1)+s}\big) |D\psi_R|^p\, dx\, d\tau. \end{aligned} \tag{2.10}$$

Because $u_k \in L^\infty(R^N \times (h,T)) \cap L^1(S_T)$, $m(p-1)-1>0$,

$$\lim_{R\to\infty}\int_h^T\int_{R^N}\big(u_k^{s+m+p-1}+u_k^{m(p-1)+s}\big)|D\psi_R|^p\,dx\,d\tau=0. \tag{2.11}$$

Let $R\to\infty$, $h\to 0$ in (2.10),

$$\sup_{0<t<T}\int_{R^N}\int_0^{u_k(x,t)}\frac{z^s}{1+z^s}\,dz\,dx+\iint_{S_T}\frac{u_k^{s-m}}{(1+u_k^s)^2}\big|Du_k^m\big|^p\,dx\,d\tau\leqslant c\int_{R^N}u_0\,dx. \tag{2.12}$$

Thus

$$\sup_{0<t<T}\int_{B_{2R}}u_k(x,t)\,dx+\int_0^T\int_{B_{2R}}\frac{u_k^{s-m}}{(1+u_k^s)^2}\big|Du_k^m\big|^p\,dx\,d\tau\leqslant c(R). \tag{2.13}$$

Let

$$u_1=\max\big\{u_k(x,t),1\big\},\qquad w=u_1^{\frac{m(p-1)-s}{p}}.$$

By Sobolev's imbedding inequality (see [11]), for $\xi\in C_0^1(B_{2R})$, $\xi\geqslant 0$, we have

$$\left(\int_{R^N}\xi^p w^r\,dx\right)^{\frac{1}{r}}\leqslant c\left[\int_{R^N}\big|D(\xi w)\big|^p\,dx\right]^{\frac{s}{p}}\left[\int_{B_{2R}}w^{\frac{p}{m(p-1)-s}}\,dx\right]^{\frac{(1-\theta)[m(p-1)-s]}{p}},$$

where

$$\theta=\left[\frac{m(p-1)-s}{p}-\frac{1}{r}\right]\left[\frac{1}{N}-\frac{1}{p}+\frac{m(p-1)-s}{p}\right]^{-1},\qquad r=\frac{p[m(p-1)+\frac{p}{N}-s]}{m(p-1)-s}.$$

It follows that

$$\iint_{S_T}\xi^p w^r\,dx\,dt\leqslant c\iint_{S_T}\big|D(\xi w)\big|^p\,dx\,dt\sup_{t\in(0,T)}\left(\int_{B_{2R}}w^{\frac{p}{m(p-1)-s}}\,dx\right)^{\frac{(r-p)[m(p-1)-s]}{p}}, \tag{2.14}$$

where $S_T=R^N\times(0,T)$. Since

$$|Dw|^p\leqslant c\frac{u_k^{s-m}}{(1+u_k^s)^2}\big|Du_k^m\big|^p\quad\text{a.e. on } \{u_k\geqslant 1\}\quad\text{and}\quad |Dw|=0\quad\text{on } \{u_k\leqslant 1\},$$

we have

$$\begin{aligned}\iint_{S_T}\big|D(\xi w)\big|^p\,dx\,dt&\leqslant c\iint_{S_T}\big(\xi^p|Dw|^p+w^p|D\xi|^p\big)\,dx\,dt\\&\leqslant c\left[\iint_{S_T}|D\xi|^p u_1^{m(p-1)-s}\,dx\,dt+\int_0^T\int_{B_{2R}}\frac{u_k^{s-m}}{(1+u_k^s)^2}\big|Du_k^m\big|^p\,dx\,dt\right].\end{aligned} \tag{2.15}$$

Hence, by (2.14), (2.15) and (2.13), we get

$$\iint_{S_T}\xi^p u_1^{m(p-1)+\frac{p}{N}-s}\,dx\,dt\leqslant c\big(s,R,|u_0|_{L^1}\big)\left(1+\iint_{S_T}|D\xi|^p u_1^{m(p-1)-s}\,dx\,dt\right).$$

Let $\xi=\psi_R^b$, ψ_R be the function satisfying (2.6) and $b=\frac{N[m(p-1)+\frac{p}{N}-s]}{p}$. Then

$$\iint_{S_T}\psi_R^{pb}u_1^{m(p-1)+\frac{p}{N}-s}\,dx\,dt\leqslant c\big(s,R,|u_0|_{L^1}\big)\left(1+\iint_{S_T}\psi_R^{pb}u_1^{m(p-1)+\frac{p}{N}-s}\,dx\,dt\right)^{\frac{m(p-1)-s}{m(p-1)-s+\frac{p}{N}}};$$

by the Morse interaction technique, the above inequality implies that (2.4) is true. $\square$

Let $Q_\rho = B_\rho(x_0) \times (t_0 - \rho^p, t_0)$ with $t_0 > (2\rho)^p$ and $u_{k1} = \max\{u_k, 1\}$. Similar to [4], also by the Morse interaction technique, we have

Lemma 2.2. *u_k satisfies*

$$\sup_{Q_\rho} u_k \leqslant c(\rho, s_1)\left(\iint_{Q_{2\rho}} u_{k1}^{m(p-1)-1+s_1}\,dx\,dt\right)^{1/s_1}, \tag{2.16}$$

where $c(\rho, s_1)$ depends on ρ and s_1, and s_1 can be any number satisfying $0 < s_1 < 1 + \frac{p}{N}$.

Lemma 2.3. *u_k satisfies*

$$\int_\tau^T\int_{B_R} \left|Du_k^m\right|^p dx\,dt \leqslant c(\tau, R), \qquad \int_\tau^T\int_{B_R} |u_{kt}|^p\,dx\,dt \leqslant c(\tau, R). \tag{2.17}$$

Proof. By Lemmas 2.1 and 2.2, $\{u_k\}$ are uniformly bounded on every compact set $K \subset S_T$. Let ψ_R be a function satisfying (2.6) and $\xi \in C_0^1(0, T)$ with $0 \leqslant \xi \leqslant 1$, $\xi = 1$ if $t \in (\tau, T)$. We choose $\eta = \psi_R^p \xi u_k^m$ in (2.5) to obtain

$$\begin{aligned}&\frac{1}{m+1}\int_{R^N} u_k^{m+1}(x, T)\psi_R^p\,dx + \iint_{S_T}\left|Du_k^m\right|^p\psi_R^p\xi\,dx\,dt \\ &\quad\leqslant \frac{1}{m+1}\iint_{S_T} u_k^{m+1}\xi'\psi_R^p\,dx\,dt - p\iint_{S_T} u_k^m\left|Du_k^m\right|^{p-2}Du_k^m\cdot D\psi_R\psi_R^{p-1}\xi\,dx\,dt.\end{aligned} \tag{2.18}$$

Noticing

$$\begin{aligned}&\iint_{S_T} u_k^m\left|Du_k^m\right|^{p-1}|D\psi_R|\psi_R^{p-1}\xi\,dx\,dt \\ &\quad\leqslant \varepsilon\iint_{S_T}\left|Du_k^m\right|^p\psi_R^p\xi\,dx\,dt + c(\varepsilon)\iint_{S_T} u_k^{pm}|D\psi_R|^p\xi\,dx\,dt,\end{aligned} \tag{2.19}$$

by (2.18), (2.19), one knows that the first inequality of (2.17) is true.

Now we will prove the second inequality of (2.17). Let

$$v(x, t) = u_{kr}(x, t) = ru_k\big(x, r^{m(p-1)-1}t\big), \quad r \in (0, 1).$$

Then

$$v_t(x, t) = \operatorname{div}\big(\left|Dv^m\right|^{p-2}Dv^m\big) - r^{m(p-1)-p_1m}k^{-\upsilon_1}\left|Dv^m\right|^{p_1} - r^{m(p-1)-1}k^{-\upsilon}v^q, \tag{2.20}$$

$$v(x, 0) = ru_k(x, 0). \tag{2.21}$$

Noticing $m(p-1) - p_1m < 0$, $m(p-1) - q < 0$, then $r^{m(p-1)-p_1m}k^{-\upsilon_1} > k^{-\upsilon_1}$, $r^{m(p-1)-q}k^{-\upsilon} > k^{-\upsilon}$, using the argument similar to the proof of Theorem 1 in [4], one can prove

$$u_k \geqslant u_{kr}.$$

It follows that

$$\frac{u_k(x, r^{m(p-1)-1}t) - u_k(x, t)}{(r^{m(p-1)-1} - 1)t} \geqslant \frac{r-1}{(1 - r^{m(p-1)-1})t}u_k\big(x, r^{m(p-1)-1}t\big).$$

Letting $r \to 1$, we get

$$u_{kt} \geqslant -\frac{u_k}{[m(p-1)-1]t}. \tag{2.22}$$

Denote $w = t^{\gamma} u_k(x,t)$, $\gamma = \frac{1}{m(p-1)-1}$. By (2.22), $w_t \geqslant 0$. By (2.1),

$$\int_{\tau}^{T}\int_{B_{2R}} t^{-\gamma} w_t \psi_R \, dx\, dt = -\int_{\tau}^{T}\int_{B_{2R}} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\psi_R \, dx\, dt$$
$$- \int_{\tau}^{T}\int_{B_{2R}} k^{-\upsilon_1}\left|Du_k^m\right|^{p_1}\psi_R \, dx\, dt - \int_{\tau}^{T}\int_{B_{2R}} k^{-\upsilon} u_k^q \psi_R \, dx\, dt + \gamma \int_{\tau}^{T}\int_{B_{2R}} t^{-1} u_k(x)\psi_R \, dx\, dt$$
$$\leqslant \frac{\beta}{\tau}\int_{\tau}^{T}\int_{B_{2R}} u_k \, dx\, dt + \left(\int_{\tau}^{T}\int_{B_{2R}} \left|Du_k^m\right|^p dx\, dt\right)^{\frac{p-1}{p}} \left(\int_{\tau}^{T}\int_{B_{2R}} |D\psi_R|^p \, dx\, dt\right)^{\frac{1}{p}}. \tag{2.23}$$

From (2.13), (2.16) and (2.23), we obtain the second inequality of (2.17). □

Proof of Theorem 1.2. By Lemmas 2.1–2.3 and [9], there exist a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and a function v such that on every compact set $K \subset S$

$$u_{k_j} \to v \quad \text{in } C(K), \qquad Du_k^m \rightharpoonup Dv^m \quad \text{in } L^p_{\mathrm{loc}}(S_T), \qquad |u_{kt}|_{L^1_{\mathrm{loc}}(S_T)} \leqslant c.$$

Similar to what was done in the proof of Theorem 2 in [4], we can prove that u satisfies (1.1) in the sense of distribution.

We now prove $v(x,0) = c\delta(x)$. Let $\chi \in C_0^1(B_R)$. Then we have

$$\int_{R^N} u_k(x,t)\chi \, dx - \int_{R^N} \varphi_k \chi \, dx$$
$$= -\int_0^t\int_{R^N} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\chi \, dx\, ds - k^{-\upsilon_1}\int_0^t\int_{R^N} \left|Du_k^m\right|^{p_1}\chi \, dx\, ds - k^{-\upsilon}\int_0^t\int_{R^N} u_k^q \chi \, dx\, ds. \tag{2.24}$$

To estimate $\int_0^t\int_{R^N} |Du_k^m|^{p-2} Du_k^m \cdot D\chi \, dx\, ds$, without loss of the generality, one can assume that $u_k > 0$. By Hölder inequality and Lemma 2.1,

$$\left|\int_0^t\int_{R^N} \left|Du_k^m\right|^{p-2} Du_k^m \cdot D\chi \, dx\, dt\right|$$
$$\leqslant c\left[\int_0^t\int_{B_{2R}} \frac{u_k^{s-m}}{(1+u_k^s)^2}\left|Du_k^m\right|^p dx\, d\tau\right]^{\frac{p-1}{p}} \cdot \left[\int_0^T\int_{B_{2R}} \left(1+u_k^s\right)^{2(p-1)} u_k^{(p-1)(m-s)} \, dx\, d\tau\right]^{\frac{1}{p}}$$
$$\leqslant c\left[\int_0^t\int_{B_{2R}} u_{k1}^{(p-1)(m-s)} + u_{k1}^{(p-1)(s+m)} \, dx\, d\tau\right]^{\frac{1}{p}}$$
$$\leqslant c\left[\int_0^t\int_{B_{2R}} u_{k1}^{m(p-1)+\frac{p}{N}-s} \, dx\, d\tau\right]^{\frac{1}{p}-d} t^d, \tag{2.25}$$

where $s \in (0, \frac{1}{N})$, $d = \frac{1-Ns}{m(p-1)N+p-sN} < \frac{1}{p}$, $u_{k1} = \max(u_k, 1)$.

Hence from (2.24), we get

$$\left|\int_{R^N} u_k \chi \, dx - \int_{R^N} \varphi_k \chi \, dx + k^{-\upsilon_1}\int_0^t\int_{R^N} u_k^{q_1}\left|Du_k^m\right|^{p_1}\chi \, dx\, ds + k^{-\upsilon}\int_0^t\int_{R^N} u_k^q \chi \, dx\, ds\right|$$
$$\leqslant \left|\int_{R^N} u_k \chi \, dx - \int_{R^N} \varphi_k \chi\left(k^{-1}x\right) dx + \int_0^{N\mu t}\int_{R^N} \left|Du_k^m\right|^{p_1}\chi\left(k^{-1}x\right) dx\, d\tau\right| + \left|\int_0^{N\mu t}\int_{R^N} u_k^q \chi\left(k^{-1}x\right) dx\, d\tau\right| \leqslant ct^d. \tag{2.26}$$

Letting $k \to \infty$, $t \to 0$ in turn, we obtain

$$\lim_{t\to 0}\int\limits_{R^N} v\chi\,dx = \chi(0)\left(\int\limits_{R^N}\varphi\,dx - \int\limits_0^\infty\int\limits_{R^N}\left|Du_k^m\right|^{p_1}dx\,dt\right) - \int\limits_0^\infty\int\limits_{R^N} u^q\,dx\,dt.$$

Thus

$$v(x,0) = c\delta(x), \qquad c = \int\limits_{R^N}\varphi\,dx - \int\limits_0^\infty\int\limits_{R^N}\left|Du_k^m\right|^{p_1}dx\,dt - \int\limits_0^\infty\int\limits_{R^N} u^q\,dx\,dt.$$

$v(x,t)$ is a solution of (1.3)–(1.4). By the assumption on uniqueness of solution, we have $v(x,t) = E_c(x,t)$ and the entire sequence $\{u_k\}$ converges to E_c as $k \to \infty$. Set $t = 1$. Then

$$u_k(x,1) = k^N u\big(kx, k^{N\mu}\big) \to E_c(x,1)$$

uniformly on every compact subset of R^N. Thus, by writing $kx = k'$, $k^{N\mu} = t'$, and dropping the prime again, we see that

$$t^{\frac{1}{\mu}} u(x,t) \to E_c\big(xt^{\frac{1}{N\mu}}, 1\big) = t^{\frac{1}{\mu}} E_c(x,t)$$

uniformly on the sets $\{x \in R^N\colon\ |x| \leqslant at^{\frac{1}{N\mu}}\}$, $a > 0$. Thus Theorem 1.2 is true. □

3. Proofs of Theorems 1.3 and 1.4

Let u be a solution of (1.1)–(1.2) and $u_k(x,t) = k^\delta u(kx, k^\sigma t)$, $k > 0$. If $\delta = \frac{q-p}{m(p-1)}$, $\sigma = \frac{q[m(p-1)-1]+p}{m(p-1)}$, then

$$u_{kt} = \operatorname{div}\big(\left|Du_k^m\right|^{p-2} Du_k^m\big) - k^{\delta+\sigma-(m\delta+1)p_1}\left|Du_k^m\right|^{p_1} - u_k^q, \tag{3.1}$$

$$u_k(x,0) = \varphi_k(x) = k^\delta\varphi(kx). \tag{3.2}$$

Lemma 3.1. *The solution u_k of* (3.1)–(3.2) *satisfies*

$$u_k(x,t) \leqslant C^* t^{-\frac{1}{q-1}}, \qquad C^* = \left(\frac{1}{q-1}\right)^{\frac{1}{q-1}}. \tag{3.3}$$

Proof. We consider the regularized problem of (3.1), say,

$$u_{kt} = \operatorname{div}\big[\big(\left|Du_k^m\right|^2 + \varepsilon\big)^{\frac{p-2}{2}} Du_k^m\big] - k^{\delta+\sigma-(m\delta+1)p_1}\left|Du_k^m\right|^{p_1} - u_k^q. \tag{3.4}$$

By the assumption of the uniqueness of the solution of (3.1)–(3.2), we can prove that

$$u_{k\varepsilon} \to u_k \quad \text{as } \varepsilon \to 0 \text{ in } C(K)$$

on every compact set $K \subset S$, where $u_{k\varepsilon}$ are the solutions of (3.4)–(3.2). By computation, it is easy to show that $C^*(t-t_0)^{-\frac{1}{q-1}}$ is a solution of (3.4) in $R^N \times (t_0, \infty)$, $t_0 > 0$. For any $\delta_1 > 0$, we choose $\delta_0 \in (0, \delta_1)$ such that

$$\left|u_{k\varepsilon}(x,\delta_1)\right|_{L^\infty(R^N)} \leqslant C^*(\delta_1 - \delta_0)^{-\frac{1}{q-1}}.$$

Hence, by the comparison principle, we have

$$u_{k\varepsilon}(x,t) \leqslant C^*(t-t_0)^{-\frac{1}{q-1}}, \quad t > \delta_1.$$

The proof of Lemma 3.1 is completed by letting $\delta_1 \to 0$ and $\varepsilon \to 0$. □

Lemma 3.2. *u_k satisfies*

$$\int\limits_\tau^T\int\limits_{B_R}\left|Du_k^m\right|^p dx\,dt \leqslant c(\tau,R), \quad \int\limits_\tau^T\int\limits_{B_R}|u_t|\,dx\,dt \leqslant c(\tau,R), \tag{3.5}$$

where $\tau \in (0,T)$.

The proof of Lemma 3.2 is similar to that of Lemma 2.3.

Proof of Theorem 1.3. By Lemma 3.1, $\{u_k\}$ are uniformly bounded on every compact set of S. Hence by [9], there exist a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that, for every compact set $K \subset S$,

$$u_{k_j} \to U \quad \text{in } C(K)$$

and

$$U(x,t) \leqslant C^* t^{-\frac{1}{q-1}}.$$

We now prove that $U(x,t) = C^* t^{-\frac{1}{q-1}}$. Let us introduce the function

$$\varphi_k^A = \min\{\varphi_k, A\} \tag{3.6}$$

and denote by $V_{K\varepsilon}^A$ the solution of (3.4) with initial value (3.6). By the comparison principle,

$$V_{K\varepsilon}^A \leqslant u_{k\varepsilon}, \tag{3.7}$$

where $u_{k\varepsilon}$ is the solution of (3.4)–(3.2).

Define

$$V_A = C^*\left(t + \frac{A^{1-q}}{q-1}\right)^{-\frac{1}{q-1}},$$

which is the solution of (3.4) with initial value

$$V_A(x,0) = A. \tag{3.8}$$

Noticing

$$\lim_{k\to\infty} \varphi_k^A(x) = \lim_{k\to\infty} \min\left\{A, \frac{\varphi(kx)|kx|^\alpha k^{\delta-\alpha}}{|x|^\alpha}\right\} = A,$$

by the uniqueness of solution of (3.4)–(3.8), we can prove (see [11])

$$V_{k\varepsilon}^A \to V_A \quad \text{as } k \to \infty \text{ in } C(K),$$

where K is a compact set in S. Moreover, by [9] and [4]

$$V_{k\varepsilon}^A \to V_k^A, \qquad u_{k\varepsilon} \to u_k \quad \text{as } k \to \infty \text{ in } C(K),$$

uniformly in K, where V_k^A is the solution of (1.1) with initial value (3.6). It follows that

$$V_k^A \to V_A \quad \text{as } k \to \infty \text{ in } C(K).$$

Letting $\varepsilon \to 0$ and $k \to \infty$ in turn in (3.7), we get

$$V_A(x,t) \leqslant V_\infty(x,t) = C^* t^{-\frac{1}{q-1}} \quad \text{in } S.$$

Since the lower bound holds for every $A > 0$, we conclude that

$$U(x,t) = V_\infty(x,t) = C^* t^{-\frac{1}{q-1}} \quad \text{in } S.$$

Thus

$$k^\delta u\big(kx, k^\beta t\big) \to C^* t^{-\frac{1}{q-1}} \quad \text{as } k \to \infty.$$

Set $t = 1$. Then

$$k^\delta u\big(kx, k^\beta\big) \to C^* \quad \text{as } k \to \infty,$$

uniformly on every compact subset on R^N. Therefore, if we set $kx = x'$, $k^\beta = t'$, and omit the primes, we obtain

$$t^{\frac{1}{q-1}} u(x,t) \to C^* \quad \text{as } t \to \infty,$$

uniformly on sets $\{x \in R^N\colon\ |x| \leqslant \alpha t^{\frac{1}{\beta}}\}$ with $\alpha > 0$ for $t > 0$ and so Theorem 1.3 is proved. $\square$

Proof of Theorem 1.4. By Lemma 3.1 and [9], there exist a subsequence $\{u_{k_j}\}$ and a function $U \in C(S)$ such that

$$u_{k_j} \to U \quad \text{in } C(K). \tag{3.9}$$

By Lemma 3.2, we can prove that U satisfies (1.1) in the sense of distribution in a manner similar to Theorem 2 of [4].

By the way, similar to [15], one is able to prove that the function U in (3.9) satisfies

$$U \in C\big(\overline{S} \setminus (0,0)\big), \qquad U(x,0) = 0, \quad \text{if } x \neq 0,$$

and

$$\lim_{t\to 0} \int_{B_R} U(x,t)\,dx = +\infty, \quad \text{for any } R > 0. \qquad \square$$

4. Some open problems

The main results of our paper (Theorems 1.2–1.4) are based on the assumption of that the solution for the Cauchy problem of (1.1) is unique. Though, for some special cases, the uniqueness problem of (1.1) had been solved as we had narrated in Remark 1.5, generally this is an open problem. Even for the simpler case such as (1.6), if the initial value $u_0(x)$ is only a bounded measure, the uniqueness problem is still open. The main difficulty comes from the term $-|Du^m|^{p_1}$, it leads the general methods used in [4,5,10,16] are invalid.

Another important problem is the existence of the similar solution and the (very) singular solution of (1.1). As we have said in Section 1, if

$$q = \frac{mp_1}{p - p_1}, \tag{4.1}$$

the problem had been partly solved in [17]. But if (4.1) is not true, it seems very difficult to solve the problem – an obvious reason is in that we are not able to get an ordinary equation similar to (1.11) now.

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